Sep. 6.

Q1 $F(x,y) = x^3 - y^3 + 9xy$, find the critical points first.

FOC: $[x]$: $3x^2 + 9y = 0$ $\Rightarrow$ $x^2 + 3y = 0$ $\Rightarrow$ $\frac{1}{9}y^4 + 3y = 0$

$[y]$: $-3y^2 + 9x = 0$ $\quad$ $y^2 = 3x$. $x = \frac{1}{3}y^2$

Solve $\Rightarrow y(y^3 + 27) = 0 \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \quad \begin{cases} y = -3 \\ x = 3 \end{cases}$

two critical points

Next, check the SOC sufficient conditions

$$H = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix},$$

where $F_{xx} = 9x$, $F_{xy} = 9$ $\quad F_{yx} = 9 \quad F_{yy} = -9y$

$$\Rightarrow H = \begin{pmatrix} 9x & 9 \\ 9 & -9y \end{pmatrix}$$

At $(x=0, y=0)$, $H = \begin{pmatrix} 0 & 9 \\ 9 & 0 \end{pmatrix}$

the first order leading principal minor: $H_1 = 0$

the second order leading principal minor $H_2 = -81 \leq 0$

So, this Hessien matrix is indefinite, non of the critical points are extremums.

Q2. $f = x_1^2+2x_1x_2-x_2^2$ $\quad g = x_1+x_2$

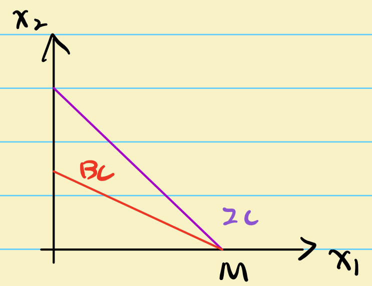
$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$ bordered Hessian matrix,

$|H_3| = 2+2-2+2 = 4 > 0$ $\quad$ compare with $(-1)^{n=2} > 0$

$|H_2| = -1 < 0$ $\quad$ alternate in sign.

$\Rightarrow$ Negative definite.

Q3.



$\Rightarrow$ we know the optimal bundle is $(M, 0)$

Solve with Kuhn-Tucker Lagrangian.

$\mathcal{L} = x_1+x_2 - \lambda(x_1+2x_2-M) + u_1x_1 + u_2x_2$

KKTs

① FOCs: $\quad 1-\lambda+u_1=0$ ①

$\quad 1-2\lambda+u_2=0$ ②

② $\quad \lambda(x_1+2x_2-M)=0$

$\lambda \geq 0$ ③ $\quad x_1+2x_2-M \leq 0$ ④

$u_1 \geq 0$ ⑤ $\quad x_1 \geq 0$ ⑥ $\quad u_1x_1=0$ ⑦

$u_2 \geq 0$ ⑧ $\quad x_2 \geq 0$ ⑨ $\quad u_2x_2=0$ ⑩

Then check for which constraints are binding.

1. Suppose $x_1+2x_2-M \leq 0$ is not binding $\Rightarrow x_1+2x_2-M<0 \Rightarrow \lambda=0$

According to ①, $u_1=-1<0$, contradicts with ⑤.

Then, we know $x_1+2x_2-M=0$. (binding)

1. Suppose. $x_2 \geq 0$ is not binding $\Rightarrow x_2>0 \Rightarrow u_2=0$

According to ②, we solve $\lambda = \frac{1}{2}$.

Then, solve $u_1 = -\frac{1}{2}$ according to ①, contradicts with ⑤.
Then, we know $x_2 = 0$ and $u_2 > 0$.

II. Since $x_2 = 0$, from $x_1 + 2x_2 = M$, we solve $x_1 = M > 0$, and $u_1 = 0$
According to ①, $\lambda = 1$, then from ②, $u_2 = 1$
Hence. $(x_1 = M,\ x_2 = 0,\ \lambda = 1,\ u_2 = 1,\ u_1 = 0)$
This satisfies all the KKT conditions.

Next, we can check the SOC.
We know two constraints are binding $\begin{cases} x_1 + 2x_2 = M \\ x_2 = 0 \end{cases}$

The bordered Hessian matrix is

$$\bar{H} = \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$

The second-order conditions requires that the $|H|$ has the same sign as $(-1)^2$, SOC is satisfied.

Envelope theorem.

let f be a continuously differentible function of $(n+k)$ variables

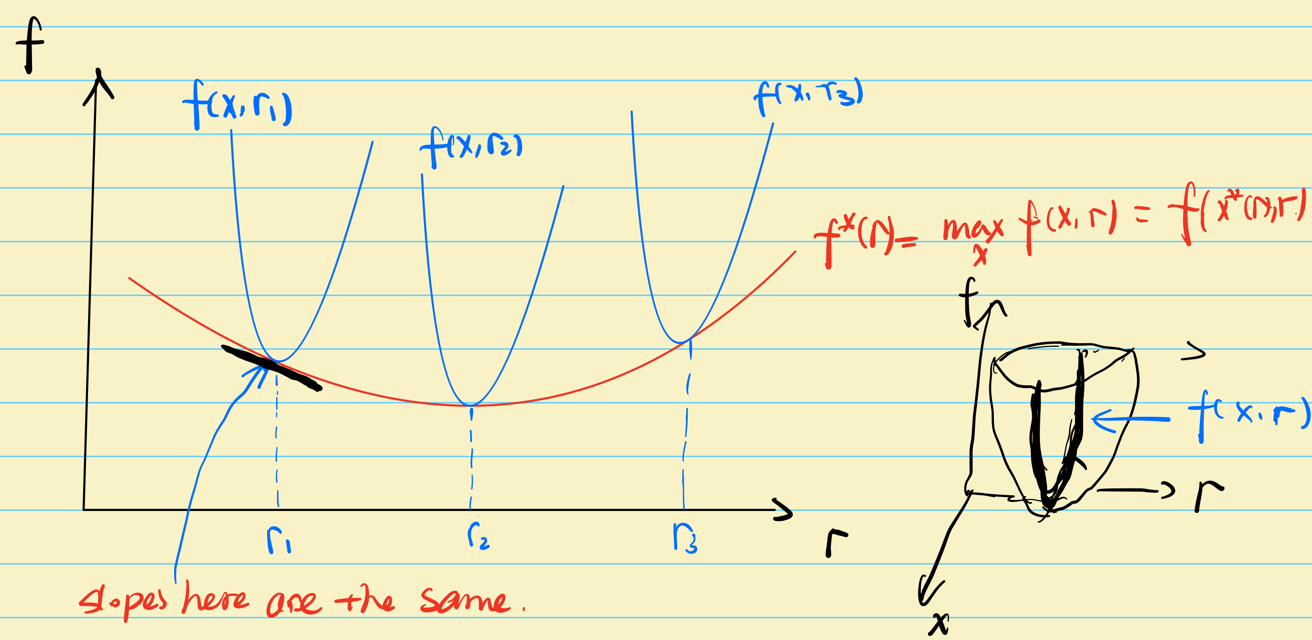
Define f^* the function of "k" variables

$$f^*(r) = \max_x f(x(r), r) = f(x^*(r), r)$$

where x is considered as endogeneous variables → n-vector
and r is considered as exogeneous variables → k-vector

The envelope theorem says if the solution of the maximization problem is a continuously differentible function of r, then

$$f^*_h(r) = f_{n+h}(x^*(r), r)$$



Application: the economic meaning of the lagrangian multiplier.

Suppose a utility function takes the form as

$$u = u(x)$$

the budget constraint is $g(x) \leq c$.

$$\max_{x,\lambda} \mathcal{L}(x, \lambda, c) = u(x) - \lambda(g(x) - c)$$

By solving the UMP. we know x is a function of c

x is typically called the Marshallian demand.

we know in the optimal condition $g(x^*) = c$

and By envelope theorem

$$\frac{\partial \mathcal{L}}{\partial c} = \lambda = \frac{\partial u(x^*(c))}{\partial c}$$

then, λ is the implicit price of income